

Problem 31) Since $h(x)$ is defined over the interval $[0, L]$, and since the desired Fourier cosine series is an even function of x whose period is $2L$, we must first extend $h(x)$ over the interval $[-L, +L]$ in such a way as to create an even function $\underline{h}(x)$. This is done by defining $\underline{h}(x) = h(|x|)$. We then proceed to determine the ordinary Fourier series of the function $\underline{h}(x)$, as follows:

$$\begin{aligned} \underline{H}(s) = \mathcal{F}\{\underline{h}(x)\} &= \int_{-L}^L \underline{h}(x) \exp(-i2\pi sx) dx = \int_{-L}^0 h(-x) \exp(-i2\pi sx) dx + \int_0^L h(x) \exp(-i2\pi sx) dx \\ &= \int_0^L h(x) \exp(+i2\pi sx) dx + \int_0^L h(x) \exp(-i2\pi sx) dx = 2 \int_0^L h(x) \cos(2\pi sx) dx. \end{aligned}$$

When $\underline{h}(x)$ is turned into a periodic function with period $2L$ (using convolution with a comb function), its Fourier series coefficients become

$$\underline{c}_n = \frac{1}{2L} \underline{H}\left(\frac{n}{2L}\right) = \frac{1}{L} \int_0^L h(x) \cos(n\pi x/L) dx.$$

Taking advantage of the fact that $\underline{c}_{-n} = \underline{c}_n$, we may now write the Fourier series of $\underline{h}(x)$ over the interval $[-L, +L]$ as follows:

$$\begin{aligned} \underline{h}(x) &= \sum_{n=-\infty}^{\infty} \underline{c}_n \exp[i2\pi nx/(2L)] = \underline{c}_0 + \sum_{n=1}^{\infty} \underline{c}_n [\exp(i\pi nx/L) + \exp(-i\pi nx/L)] \\ &= \underline{c}_0 + \sum_{n=1}^{\infty} 2\underline{c}_n \cos(n\pi x/L). \end{aligned}$$

Comparison with the Fourier cosine series expansion of $h(x)$ over the interval $[0, L]$ now yields

$$\begin{aligned} c_0 = \underline{c}_0 &= \frac{1}{L} \int_0^L h(x) dx, \\ c_n = 2\underline{c}_n &= \frac{2}{L} \int_0^L h(x) \cos(n\pi x/L) dx. \end{aligned}$$
